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STOCHASTIC MODELING OF THE TIME-AVERAGED EQUATIONS FOR CLIMATE DYNAMICS

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Abstract

We report on the results of the first year's activities under NASA grant 5254, covering a period from March 15th 1978 to March 14th 1979, "Stochastic modeling of the time-averaged equations for climate dynamics". We discuss two analyses, both based on a simplified set of barotropic equations for representing large scale non-linear atmospheric circulation characteristics.

We have essentially completed study of the statistical properties of this set of equations for small values of a parameter k_o, corresponding to the physical case of large zonal westerlies and relatively weak eddy motions. We have also started an investigation of the effect of seasonal type forcing on the solution of these equations, and some preliminary numerical results are presented.

In addition to describing the formal analyses, we include a more general discussion on the significance of the results in the context of the use of the simplified equations, and indicate the directions along which we expect progress to be made.

Introduction

The first year's work performed under NASA grant 5254 has concentrated on study of the statistical properties of a particularly simple model for large scale atmospheric motions - a three wave component barotropic system, first introduced in the meteorological literature by Lorenz[1], the so called "minimum hydrodynamic system" of equations. The immediate objective of our program was to investigate the long term statistical properties of this model, employing time-averaged quantities as the dependent state variables. Introduction of the latter, it was hoped, could lead to direct means for obtaining detailed climatic states more efficiently than by computing average statistics from long term integration of deterministic general circulation models.

Although such a basic approach to climate is appealing on fundamental grounds, it leads to considerable analytic complexity because of the multiplication in the number of differential equations that result from the time-averaging procedure; we are in fact confronted with the need to deal with a formally infinite number of coupled non-linear differential equations defining the statistical moments of the system. It was expected that the problem of closure of this set could be tackled more easily by using the simplified minimum equations as a model for the more complete and realistic system, not just because of ease of computation of numerical solutions, but primarily because analytic solutions for the minimum equations are available, and these can be used as a check on approximate methods of closure of the moment equation set.

One approximation method that held promise (based in part on experience gained with the same system of equations applied to weather forecasting problems[2]) was a linearisation of the moment equations and truncation at a fairly low order for estimation of small climatic changes. A large part of the first year's work

stemmed from this approach, but, as we shall describe shortly, during the course of the investigations an important property of the minimum equations was discovered that largely eliminates the need to consider a small climatic change linearisation approximation for closure.

A second theme, introduction of seasonal variation into the study of climate statistics, was also started. Here again we have used the minimum equation system as a model and simulated seasonality by introduction of additional forcing terms in the equations.

Closure of the moment equations for the minimum equations

In the process of calculating magnitudes of higher order moments of the minimum equation solutions, an unexpected characteristic was discovered, namely a systematic reduction in magnitude of the moments with increasing order, thus enabling us to formulate a closure technique for the infinite set of moment equations that we can demonstrate gives good approximations to exact solutions. This result applies for small values of the parameter k_o (to be defined below) corresponding to the case of relatively large Westerlies and low eddy energy, matching conditions that Lorenz originally suggested best approximated mid-latitude conditions on the earth. We have completed a comprehensive analysis of the reasons for this useful result, and present next a summary of the approach and the analytic results.

The equations governing the temporal development of the amplitudes of the three largest scale Fourier components of the barotropic flow in a rectangular domain have been shown by Lorenz to be

$$\begin{aligned} \dot{A}_1 &= c_1 A_2 A_3 \\ \dot{A}_2 &= c_2 A_3 A_1 \\ \dot{A}_3 &= c_3 A_1 A_2 \end{aligned} \tag{1}$$

where

$$c_1 = 1/\alpha(\alpha^2 + 1)$$
$$c_2 = \alpha^3/(\alpha^2 + 1) \qquad (2)$$
$$c_3 = (1 - \alpha^2)/2\alpha$$

and $\alpha = k/\ell =$ the ratio of the meridional to zonal dimensions of the rectangular domain. We note that the important Coriolis grandient effects are omitted in this model. The time averaged form of the moment equations can be shown to be

$$\dot{\mu}_1 = c_1(\mu_2\mu_3 + \sigma_{23})$$
$$\dot{\mu}_2 = c_2(\mu_3\mu_1 + \sigma_{31})$$
$$\dot{\mu}_3 = c_3(\mu_1\mu_2 + \sigma_{12})$$

$$\dot{\sigma}_{11} = 2c_1(\mu_2\sigma_{13} + \mu_3\sigma_{12}) + 2c_1\tau_{123}$$
$$\dot{\sigma}_{22} = 2c_2(\mu_3\sigma_{21} + \mu_1\sigma_{23}) + 2c_2\tau_{123}$$
$$\dot{\sigma}_{33} = 2c_3(\mu_1\sigma_{32} + \mu_2\sigma_{31}) + 2c_3\tau_{123}$$
$$\dot{\sigma}_{12} = c_1(\mu_2\sigma_{23} + \mu_3\sigma_{22}) + c_2(\mu_1\sigma_{13} + \mu_3\sigma_{11}) + c_1\tau_{223} + c_2\tau_{113}$$
$$\dot{\sigma}_{23} = c_2(\mu_3\sigma_{31} + \mu_1\sigma_{33}) + c_3(\mu_2\sigma_{21} + \mu_1\sigma_{22}) + c_2\tau_{331} + c_3\tau_{221}$$
$$\dot{\sigma}_{31} = c_3(\mu_1\sigma_{12} + \mu_2\sigma_{11}) + c_1(\mu_3\sigma_{32} + \mu_2\sigma_{33}) + c_3\tau_{112} + c_1\tau_{332}$$
$$\dot{\tau}_{111} = 3c_1(\mu_2\tau_{113} + \mu_3\tau_{112} + \lambda_{1123} - \sigma_{23}\sigma_{11}) \qquad (3)$$
$$\dot{\tau}_{222} = 3c_2(\mu_3\tau_{221} + \mu_1\tau_{223} + \lambda_{2231} - \sigma_{31}\sigma_{22})$$
$$\dot{\tau}_{333} = 3c_3(\mu_1\tau_{332} + \mu_2\tau_{331} + \lambda_{3312} - \sigma_{12}\sigma_{33})$$
$$\dot{\tau}_{112} = 2c_1(\mu_2\tau_{123} + \mu_3\tau_{122} - \sigma_{23}\sigma_{12} + \lambda_{1223}) + c_2(\mu_1\tau_{113} + \mu_3\tau_{111} - \sigma_{13}\sigma_{11} + \lambda_{1113})$$
$$\dot{\tau}_{223} = 2c_2(\mu_3\tau_{123} + \mu_1\tau_{233} - \sigma_{31}\sigma_{23} + \lambda_{2331}) + c_3(\mu_2\tau_{221} + \mu_1\tau_{222} - \sigma_{21}\sigma_{22} + \lambda_{2221})$$
$$\dot{\tau}_{331} = 2c_3(\mu_1\tau_{123} + \mu_2\tau_{311} - \sigma_{12}\sigma_{31} + \lambda_{3112}) + c_1(\mu_3\tau_{332} + \mu_2\tau_{333} - \sigma_{32}\sigma_{33} + \lambda_{3332})$$
$$\dot{\tau}_{113} = 2c_1(\mu_3\tau_{123} + \mu_2\tau_{133} - \sigma_{23}\sigma_{13} + \lambda_{1233}) + c_3(\mu_1\tau_{112} + \mu_2\tau_{111} - \sigma_{12}\sigma_{11} + \lambda_{1112})$$
$$\dot{\tau}_{221} = 2c_2(\mu_1\tau_{123} + \mu_3\tau_{211} - \sigma_{31}\sigma_{21} + \lambda_{2311}) + c_1(\mu_2\tau_{223} + \mu_3\tau_{222} - \sigma_{23}\sigma_{22} + \lambda_{2223})$$
$$\dot{\tau}_{331} = 2c_3(\mu_2\tau_{231} + \mu_1\tau_{322} - \sigma_{12}\sigma_{32} + \lambda_{3122}) + c_2(\mu_3\tau_{331} + \mu_1\tau_{333} - \sigma_{31}\sigma_{33} + \lambda_{3331})$$
$$\dot{\tau}_{123} = c_1(\mu_2\tau_{233} + \mu_3\tau_{223} - \sigma_{23}^2 + \lambda_{2233}) + c_2(\mu_1\tau_{133} + \mu_3\tau_{113} - \sigma_{13}^2 + \lambda_{1133})$$
$$+ c_3(\mu_2\tau_{112} + \mu_1\tau_{122} - \sigma_{12}^2 + \lambda_{1122})$$

where the moments are defined as

$$
\begin{aligned}
\mu_i &= \langle A_i \rangle \\
\sigma_{ij} &= \langle (A_i-\mu_i)(A_j-\mu_j)\rangle = \langle A_iA_j\rangle - \mu_i\mu_j \\
\tau_{ijk} &= \langle (A_i-\mu_i)(A_j-\mu_j)(A_k-\mu_k)\rangle = \langle A_iA_jA_k\rangle - \mu_i\sigma_{jk} - \mu_j\sigma_{ik} \\
&\qquad - \mu_k\sigma_{ij} - \mu_i\mu_j\mu_k \\
\lambda_{ijkm} &= \langle (A_i-\mu_i)(A_j-\mu_j)(A_k-\mu_k)(A_m-\mu_m)\rangle \\
&= \langle A_iA_jA_kA_m\rangle - \mu_m\tau_{ijk} - \mu_k\tau_{ijm} - \mu_j\tau_{ikm} - \mu_i\tau_{jkm} \\
&\qquad - \mu_m\mu_i\tau_{jk} - \mu_m\mu_j\tau_{ik} - \mu_k\mu_i\tau_{jm} - \mu_k\mu_j\tau_{im} - \mu_k\mu_m\sigma_{ij} - \mu_i\mu_j\sigma_{km} - \mu_i\mu_j\mu_k\mu_m
\end{aligned}
\tag{4}
$$

As shown by Lorenz[1] exact analytic solutions for (1) are given by

$$
\begin{aligned}
A_1 &= A_1^* \,\mathrm{dn}\,(h(t-t^*), k_0) \\
A_2 &= A_2^* \,\mathrm{sn}\,(h(t-t^*), k_0) \qquad \alpha > 1 \\
A_3 &= A_3^* \,\mathrm{cn}\,(h(t-t^*), k_0)
\end{aligned}
\tag{5}
$$

where dn , sn , cn are elliptic functions, modulus k_o , and h , t* and k_o are given in terms of the maximum amplitudes A_1^*, A_2^*, A_3^*, by

$$
\begin{aligned}
h &= \sqrt{\frac{\alpha^2(\alpha^2-1)}{\alpha^2+1}}\; A_1^* \\
k_0^2 &= \sqrt{\frac{2}{\alpha^4(\alpha^4-1)}}\; \frac{A_2^* A_3^*}{A_1^{*2}} \\
-ht^* &= \int_0^{\varphi_0} \frac{d\theta}{(1-k_0^2\sin^2\theta)^{\frac{1}{2}}} \\
&= F(\varphi_0, k_0)
\end{aligned}
\tag{6}
$$

The parameter k_o determines the periodicity of the elliptic function solutions, which in turn corresponds to a vacillation phenomenon, analogous to variations in zonal index that occur in the real atmosphere. Larger k_o values corresponds to slower cycling, the periodicity becoming infinite as k_o approaches 1. Lorenz[1] chose a value of $k_o^2 = 0.2$ as representative of typical

mid-latitude conditions; this yields a vacillation period of 5.6 days and, with the parameter $\alpha = 2$, a zonal wave length for the Fourier component of 5000 km.

Exact analytic results can be calculated from (1) for moments, defined over an averaging time T. We list values up to the third order moments:

$$\mu_1 = \frac{A_1^*}{hT}\int_{h(t-t^*)}^{h(T+t-t^*)} dn(u,k_0)\,du = \frac{A_1^*}{hT}\,\mathrm{am}\,u\Big|_b^a$$

$$\mu_2 = \frac{A_2^*}{hTk_0}\ln(dn\,u - k_0\,cn\,u)\Big|_a^b$$

$$\mu_3 = \frac{A_3^*}{hTk_0}\arcsin(k_0\,\mathrm{sin}\,u)\Big|_a^b$$

$$\sigma_{12} = -\frac{A_1^*A_2^*}{hT}\,cn\,u\Big|_a^b - \mu_1\mu_2$$

$$\sigma_{13} = \frac{A_1^*A_3^*}{hT}\,sn\,u\Big|_a^b - \mu_1\mu_3$$

$$\sigma_{23} = \cdot\frac{A_2^*A_3^*}{hT}\,dn\,u\Big|_a^b - \mu_2\mu_3$$

$$\sigma_{11} = \frac{A_1^{*2}}{hT}\,\mathbb{E}(\mathrm{am}\,u, k_0)\Big|_a^b - \mu_1^2$$

$$\sigma_{22} = \frac{A_2^{*2}}{hTk_0^2}\Big[u - \mathbb{E}(\mathrm{am}\,u, k_0)\Big]_a^b - \mu_2^2$$

$$\sigma_{33} = \frac{A_3^{*2}}{hTk_0^2}\Big[\mathbb{E}(\mathrm{am}\,u, k) - (1-k_0^2)u\Big]_a^b - \mu_3^2$$

$$\tau_{111} = \frac{A_1^{*3}}{hT}\Big[\frac{k_0^2}{2}\,sn\,u\,cn\,u + \frac{1}{2}(2-k_0^2)\,\mathrm{am}\,u\Big]_a^b - 3\mu_1\sigma_{11} - \mu_1^3$$

$$\tau_{222} = \frac{A_2^{*3}}{2k_0^2hT}\cdot\Big[cn\,u\,dn\,u + \frac{1+k_0^2}{k_0}\ln(dn\,u - k_0\,cn\,u)\Big]_a^b - 3\mu_2\sigma_{22} - \mu_2^3 \qquad (7)$$

$$\tau_{333} = \frac{A_3^{*3}}{2k_0^2hT}\Big[sn\,u\,dn\,u - \frac{1-2k_0^2}{k_0}\arcsin(k_0\,sn\,u)\Big]_a^b - 3\mu_3\sigma_{33} - \mu_3^3$$

$$\tau_{112} = \frac{A_2^*A_1^{*2}}{hT}\Big[\frac{1-k_0^2}{2k_0}\ln(dn\,u - k_0\,cn\,u) - \frac{1}{2}cn\,u\,dn\,u\Big]_a^b - 2\mu_1\sigma_{12} - \mu_2\sigma_{11} - \mu_1^2\mu_2$$

$$\tau_{113} = \frac{A_1^{*2} A_3^*}{hT}\left[\frac{1}{2k_0}\arcsin(k_0 \,\mathrm{sn}\,u) + \frac{1}{2}\,\mathrm{sn}\,u\,\mathrm{dn}\,u\right]_b^a - 2\mu_1\sigma_{13} - \mu_3\sigma_{11} - \mu_1^2\mu_3$$

$$\tau_{122} = \frac{A_1^{*} A_2^{*2}}{2hT}\left[\mathrm{am}\,u - \mathrm{sn}\,u\,\mathrm{cn}\,u\right]_b^a - 2\mu_2\sigma_{12} - \mu_1\sigma_{22} - \mu_2^2\mu_1$$

$$\tau_{223} = \frac{A_2^{*2} A_3^*}{hT}\left[\frac{1}{2k_0^3}\arcsin(k_0\,\mathrm{sn}\,u) - \frac{1}{2k_0^2}\,\mathrm{sn}\,u\,\mathrm{dn}\,u\right]_b^a - 2\mu_2\sigma_{23} - \mu_3\sigma_{22} - \mu_2^2\mu_3$$

$$\tau_{133} = \frac{A_3^{*2} A_1^*}{hT}\left[\frac{1}{2}\,\mathrm{sn}\,u\,\mathrm{cn}\,u + \frac{1}{2}\,\mathrm{am}\,u\right]_b^a - 2\mu_3\sigma_{13} - \mu_1\sigma_{33} - \mu_1\mu_3^2$$

$$\tau_{233} = \frac{A_3^{*2} A_2^*}{hT}\left[\frac{k_0^2-1}{2k_0^3}\ln(\mathrm{dn}\,u - k_0\,\mathrm{cn}\,u) - \frac{1}{2k_0^2}\,\mathrm{cn}\,u\,\mathrm{dn}\,u\right]_b^a - 2\mu_3\sigma_{23}$$

$$- \mu_2\sigma_{33} - \mu_3^2\mu_2$$

$$\tau_{123} = \frac{A_1^* A_2^* A_3^*}{hT}\left.\frac{\mathrm{sn}^2 u}{2}\right|_b^a - \mu_1\sigma_{23} - \mu_2\sigma_{13} - \mu_3\sigma_{21} - \mu_1\mu_2\mu_3$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

where $\mathbb{K}$ and $\mathbb{E}$ are complete elliptic integrals of the the first and second kinds.

If we let the averaging time approach infinity, a number of the moments defined by (7) vanish, and the remainder are given by

$$\mu_1 = \frac{\pi}{2}\frac{A_1^*}{\mathbb{K}}$$

$$\sigma_{11} = A_1^{*2}\left[\frac{\mathbb{E}}{\mathbb{K}} - \left(\frac{\pi}{2\mathbb{K}}\right)^2\right]$$

$$\sigma_{22} = \frac{A_2^{*2}}{k_0^2}\left(1 - \frac{\mathbb{E}}{\mathbb{K}}\right)$$

$$\sigma_{33} = \frac{A_3^{*2}}{k_0^2}\left(\frac{\mathbb{E}}{\mathbb{K}} - 1 + k_0^2\right) \qquad (8)$$

$$\tau_{133} = \frac{\pi}{4\mathbb{K}} A_1^* A_2^{*2} - \frac{\pi}{2\mathbb{K} k_0^2} A_1^* A_3^{*2} \left(\frac{\mathbb{E}}{\mathbb{K}} - 1 + k_0^2 \right)$$

$$\tau_{122} = \frac{\pi}{4\mathbb{K}} A_1^* A_2^{*2} - \frac{\pi}{2\mathbb{K} k_0^2} A_1^* A_2^{*2} \left(1 - \frac{\mathbb{E}}{\mathbb{K}} \right)$$

$$\tau_{111} = \frac{\pi}{4\mathbb{K}} A_1^{*3} (2 - k_0^2) - \frac{3\pi}{2\mathbb{K}^2} A_1^{*3} \mathbb{E} + \frac{\pi}{4\mathbb{K}^3} A_1^{*3}$$

In this case the moment equations (2) reduce to an algebraic set:

$$\begin{aligned}
&\mu_2 \mu_3 + \sigma_{23} = 0 \\
&\mu_1 \mu_3 + \sigma_{13} = 0 \\
&\mu_1 \mu_2 + \sigma_{12} = 0 \\
&\mu_2 \sigma_{13} + \mu_3 \sigma_{12} + \tau_{123} = 0 \\
&\mu_1 \sigma_{23} + \mu_3 \sigma_{12} + \tau_{123} = 0 \\
&\mu_1 \sigma_{23} + \mu_2 \sigma_{13} + \tau_{123} = 0 \\
&c_1 (\mu_2 \sigma_{23} + \mu_3 \sigma_{22} + \tau_{223}) + c_2 (\mu_1 \sigma_{13} + \mu_3 \sigma_{11} + \tau_{113}) = 0 \\
&c_2 (\mu_3 \sigma_{31} + \mu_1 \sigma_{33} + \tau_{331}) + c_3 (\mu_2 \sigma_{21} + \mu_1 \sigma_{22} + \tau_{221}) = 0 \\
&c_3 (\mu_1 \sigma_{12} + \mu_2 \sigma_{11} + \tau_{112}) + c_1 (\mu_3 \sigma_{32} + \mu_2 \sigma_{33} + \tau_{332}) = 0 \\
&\vdots
\end{aligned} \tag{9}$$

and we should be able to derive the solutions (8) from (9), once the two integral invariants of the system - energy and enstrophy - are prescribed. The amplitudes A_1^*, A_2^* and A_3^* are in fact calculable from the energy V and enstrophy E :

$$\begin{aligned}
A_1^{*2} &= \frac{2}{d^2} \left(\frac{d^2+1}{d^2} 2k^2 E - V \right) \\
A_2^{*2} &= \frac{2d^2}{d^2-1} \left(V - \frac{1}{d^2} 2k^2 E \right)
\end{aligned} \tag{10}$$

$$A_3^{*2} = \frac{\nu^2+1}{\nu^2}\left(V - \frac{1}{2\nu^2} 2k^2 E\right)$$

We have found no general technique for solving (9), given E and V, and indeed the situation is made more perplexing by the fact that some of the original differential equations (2) are no longer independent in the steady state form, (9). For example, the three time-dependent equations for the second order moments

$$\begin{aligned}
\dot{\sigma}_{11} &= 2c_1(\mu_2\sigma_{13} + \mu_3\sigma_{12} + \tau_{123}) \\
\dot{\sigma}_{22} &= 2c_2(\mu_3\sigma_{21} + \mu_1\sigma_{23} + \tau_{123}) \\
\dot{\sigma}_{33} &= 2c_3(\mu_1\sigma_{32} + \mu_2\sigma_{31} + \tau_{123})
\end{aligned} \tag{11}$$

reduce to the single equation

$$\tau_{123} = 2\mu_1\mu_2\mu_3 \tag{12}$$

when the time derivatives vanish. Thus far we see no profound reason for studying this anachronism and have not tried to understand structure of the steady state system. As we appreciate the subtlies of the statistical behavior of the minimum equations we may wish to return to the problem.

For small values of k_o^2, we can expand the elliptic function solutions of equation (8) (the infinite averaging time case), yielding

$$\begin{aligned}
\mu_1 &\approx A_1^*\left(1 - \frac{k_o^2}{4}\right) = O(1) \\
\sigma_{11} &\approx A_1^{*2}\frac{k_o^4}{32} = O(k_o^4) \\
\sigma_{22} &\simeq A_2^{*2}\left(\frac{1}{2} + \frac{1}{16}k_o^2\right) = O(1) \\
\sigma_{33} &\simeq A_3^{*2}\left(\frac{1}{2} - \frac{1}{16}k_o^2\right) = O(1)
\end{aligned}$$

$$\tau_{133} \approx A_1^* A_3^{*2}\left(\frac{k_0^2}{16} - \frac{k_0^4}{64}\right) = O(k_0^2)$$

$$\tau_{122} \approx A_1^* A_2^{*2}\left(-\frac{k_0^2}{16} + \frac{k_0^4}{64}\right) = O(k_0^2) \qquad (13)$$

$$\tau_{111} \qquad = O(k_0^6)$$

$$\vdots$$

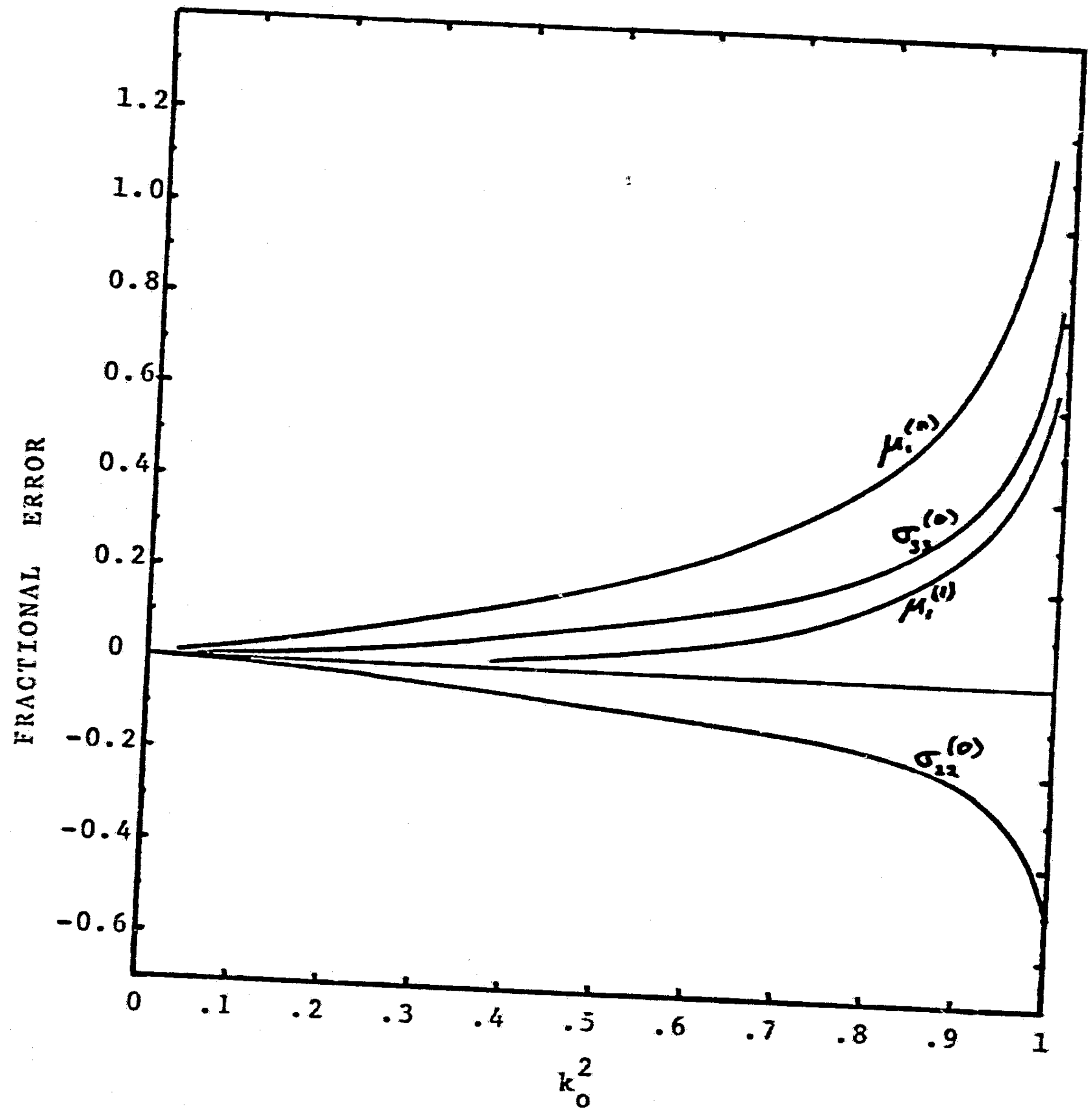
and from studying the order of magnitudes (in k_0^2) of the terms in (13) we see that the first and second order moments are in fact given to order k_0^2 by solution of the closed set of equations

$$
\begin{aligned}
&\mu_1\mu_2\mu_3 = 0\\
&\quad \sigma_{23} = -\mu_2\mu_3\\
&\quad \sigma_{13} = -\mu_1\mu_3\\
&\quad \sigma_{12} = -\mu_1\mu_2\\
&c_1(\mu_2\sigma_{23} + \mu_3\sigma_{22}) + c_2\mu_1\sigma_{13} = 0\\
&c_1(\mu_2\sigma_{33} + \mu_3\sigma_{23}) + c_3\mu_1\sigma_{12} = 0\\
&c_2(\mu_1\sigma_{33} + \mu_3\sigma_{13}) + c_3(\mu_1\sigma_{22} + \mu_2\sigma_{12}) = 0
\end{aligned} \qquad (14)
$$

with the integral constraint conditions

$$
\begin{aligned}
4k^2E &= a^2\mu_1^2 + (\sigma_{22} + \mu_2^2) + \frac{2a^2}{a^2+1}(\sigma_{33} + \mu_3^2)\\
2V &= \sigma_{22} + 2\sigma_{33} + \mu_1^2 + \mu_2^2 + 2\mu_3^2
\end{aligned} \qquad (15)
$$

Our numerical work with the minimum equations has centered on choice of $k_0^2 = 0.2$, as was the original study by Lorenz. The approximation for small k_0 that we are discussing is quite adequate to cover this situation, as can be seen in the graphs plotted in figure (1). Here we show approximate solutions for some of the moments corresponding to equation (14). We give the variation



1.2
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
FRACTIONAL ERROR
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
k_o^2
$\mu_1^{(0)}$
$\sigma_{33}^{(0)}$
$\mu_1^{(1)}$
$\sigma_{11}^{(0)}$

FIGURE I.

with k_0 of the first order approximation for μ_1 , σ_{11} and σ_{22}, and the second order approximation for μ_1.

If we go on to look at the structure of the moment solution system for finite T values, again with small values of k_0^2 , we find a somewhat more complex ordering, but still of a nature that enables us to close the moment equation set. For example, from the analytic forms (7), using the expansions

$$
\begin{aligned}
dn\,(u) &= 1 - \tfrac{1}{2} k_0^2 \sin^2 u + \cdots \\
cn\,(u) &= \cos u + \tfrac{1}{4} k_0^2 (u - \sin u \cos u) \sin u + \cdots \\
sn\,(u) &= \sin u - \tfrac{1}{4} k_0^2 (u - \sin u \cos u) \cos u + \cdots
\end{aligned}
\tag{16}
$$

for the elliptic functions, the leading orders of magnitudes of the moments are found to be given by

$$
\begin{aligned}
&\mu_1 = A_1 + O(k_0^2)\ , \mu_2 = O(1)\ , \mu_3 = O(1)\ ; \\
&\sigma_{12} = O(k_0^2)\ , \sigma_{13} = O(k_0^2)\ , \sigma_{11} = O(k_0^4) \\
&\sigma_{22} = O(1)\ , \sigma_{33} = O(1)\ , \sigma_{23} = O(1)\ ; \\
&\tau_{111} = O(k_0^6)\ , \tau_{112} = O(k_0^4), \tau_{113} = O(k_0^4)\ , \\
&\tau_{122} = O(k_0^2)\ , \tau_{133} = O(k_0^2), \tau_{123} = O(k_0^2), \\
&\tau_{222} = O(1)\ , \tau_{223} = O(1)\ , \tau_{332} = O(1)\ .
\end{aligned}
\tag{17}
$$

and satisfy the differential equations

$$
\begin{aligned}
&\dot{\mu}_2 = c_2 \mu_1 \mu_3\ , \dot{\mu}_3 = c_3 \mu_2 \mu_1\ , \dot{\mu}_1 = 0\ ; \\
&\dot{\sigma}_{22} = 2c_2 \mu_1 \sigma_{23}\ , \dot{\sigma}_{33} = 2c_3 \mu_1 \sigma_{23}\ , \dot{\sigma}_{23} = c_2 \mu_1 \sigma_{33} + c_3 \mu_1 \sigma_{22}\ , \\
&\dot{\sigma}_{11} = \dot{\sigma}_{12} = \dot{\sigma}_{13} = 0 \\
&\dot{\tau}_{222} = 3c_2 \mu_1 \tau_{223}\ , \dot{\tau}_{333} = 3c_3 \mu_1 \tau_{233}\ , \\
&\dot{\tau}_{223} = 2c_2 \mu_1 \tau_{233} + c_3 \mu_1 \tau_{222}\ , \dot{\tau}_{233} = 2c_3 \mu_1 \tau_{223} + c_2 \mu_1 \tau_{333}\ , \\
&\dot{\tau}_{112} = \dot{\tau}_{113} = \dot{\tau}_{122} = \dot{\tau}_{133} = \dot{\tau}_{123} = 0
\end{aligned}
\tag{18}
$$

– a closed set for the variables μ_i, σ_{ij} and τ_{ijk}.

A systematic procedure for developing the approximation to the moment equations for small k_0^2 can be devised by considering an expansion of the dependent variables as power series in k_0^2. The complete development is rather laborious, and we shall only outline a few results. We first simplify the algebra by introducing non-dimensional variables defined as

$$\begin{aligned} X_1 &= A_1/A_1^*, \quad X_2 = A_2/A_2^*, \quad X_3 = A_3/A_3^*, \\ t' &= th, \quad t'^* = t^* h, \\ \dot{X} &= \frac{dX}{dt'} . \end{aligned} \tag{19}$$

giving the modified form of the differential equations

$$\begin{aligned} \dot{X}_1 &= -k_0^2 X_2 X_3 \\ \dot{X}_2 &= X_3 X_1 \\ \dot{X}_3 &= -X_1 X_2 \end{aligned} \tag{20}$$

The moment equations now read

$$\begin{aligned} \dot{\mu}_1 &= -k_0^2 (\mu_2 \mu_3 + \sigma_{23}) \\ \dot{\mu}_2 &= \mu_3 \mu_1 + \sigma_{13} \\ \dot{\mu}_3 &= \mu_1 \mu_3 + \sigma_{12} \\ \dot{\sigma}_{11} &= -2k_0^2 (\mu_2 \sigma_{13} + \mu_3 \sigma_{12} + \tau_{123}) \\ \dot{\sigma}_{22} &= 2(\mu_1 \sigma_{23} + \mu_3 \sigma_{13} + \tau_{123}) \\ \dot{\sigma}_{33} &= -2(\mu_1 \sigma_{23} + \mu_2 \sigma_{13} + \tau_{123}) \\ \dot{\sigma}_{12} &= -k_0^2 (\mu_2 \sigma_{23} + \mu_3 \sigma_{22} + \tau_{223}) + (\mu_1 \sigma_{13} + \mu_3 \sigma_{11} + \tau_{113}) \\ \dot{\sigma}_{13} &= -k_0^2 (\mu_2 \sigma_{33} + \mu_3 \sigma_{23} + \tau_{233}) - (\mu_1 \sigma_{12} + \mu_1 \sigma_{11} + \tau_{112}) \\ \dot{\sigma}_{23} &= \mu_1 \sigma_{33} + \mu_3 \sigma_{13} + \tau_{133} - (\mu_1 \sigma_{22} + \mu_2 \sigma_{12} + \tau_{122}) \end{aligned} \tag{21}$$

When power series expansions for the moments are introduced into (21), a series of closed coupled equations result. We cite only the lowest order result.

Thus, with the expansions

$$
\begin{aligned}
\mu_1 &= \mu_1^{(0)} + k_0^2 \mu_1^{(1)} + k_0^4 \mu_1^{(2)} + \cdots \\
\mu_2 &= \mu_2^{(0)} + k_0^2 \mu_2^{(1)} + k_0^4 \mu_2^{(2)} + \cdots \\
\mu_3 &= \mu_3^{(0)} + k_0^2 \mu_3^{(1)} + k_0^4 \mu_3^{(2)} + \cdots \\
\sigma_{11} &= k_0^4 \sigma_{11}^{(2)} + k_0^6 \sigma_{11}^{(3)} + \cdots \\
\sigma_{12} &= k_0^2 \sigma_{12}^{(1)} + k_0^4 \sigma_{12}^{(2)} + \cdots \\
\sigma_{13} &= k_0^2 \sigma_{13}^{(1)} + k_0^4 \sigma_{13}^{(2)} + \cdots \\
\sigma_{33} &= \sigma_{33}^{(0)} + k_0^2 \sigma_{33}^{(1)} + \cdots \\
\sigma_{23} &= \sigma_{23}^{(0)} + k_0^2 \sigma_{23}^{(1)} + k_0^4 \sigma_{23}^{(2)} + \cdots \\
\sigma_{22} &= \sigma_{22}^{(0)} + k_0^2 \sigma_{22}^{(1)} + \cdots \\
\tau_{333} &= \tau_{333}^{(0)} + k_0^2 \tau_{333}^{(1)} + k_0^4 \tau_{333}^{(2)} + \cdots \\
\tau_{223} &= \tau_{223}^{(0)} + k_0^2 \tau_{223}^{(1)} + k_0^4 \tau_{223}^{(2)} + \cdots \\
\tau_{332} &= \tau_{332}^{(0)} + k_0^2 \tau_{332}^{(0)} + \cdots \\
\tau_{111} &= k_0^6 \tau_{111}^{(3)} + k_0^8 \tau_{111}^{(4)} + \cdots \\
\tau_{112} &= k_0^4 \tau_{112}^{(2)} + k_0^6 \tau_{112}^{(3)} + \cdots \\
\tau_{113} &= k_0^4 \tau_{113}^{(2)} + k_0^6 \tau_{113}^{(3)} + \cdots \\
\tau_{122} &= k_0^2 \tau_{122}^{(1)} + k_0^4 \tau_{122}^{(2)} + \cdots \\
\tau_{133} &= k_0^2 \tau_{133}^{(1)} + k_0^4 \tau_{133}^{(2)} + \cdots \\
\tau_{123} &= k_0^2 \tau_{123}^{(1)} + k_0^4 \tau_{123}^{(2)} + \cdots \\
&\vdots
\end{aligned}
\tag{22}
$$

we find that $\dot{\mu}_1^{(0)} = 0$, $\mu_1^{(0)} = 1$, and that

$$
\begin{aligned}
&\dot{\mu}_2^{(0)} = \mu_3^{(0)}, \quad \dot{\mu}_3^{(0)} = -\mu_0^{(2)}; \\
&\dot{\sigma}_{22}^{(0)} = 2\sigma_{23}^{(0)}, \quad \dot{\sigma}_{33}^{(0)} = -2\sigma_{23}^{(0)}, \quad \dot{\sigma}_{23}^{(0)} = \sigma_{33}^{(0)} - \sigma_{22}^{(0)}; \\
&\dot{\tau}_{333}^{(0)} = -3\tau_{332}^{(0)}, \quad \dot{\tau}_{222}^{(0)} = 3\tau_{223}^{(0)}, \quad \dot{\tau}_{322}^{(0)} = 2\tau_{223}^{(0)} - \tau_{222}^{(0)}, \quad \dot{\tau}_{233}^{(0)} = 2\tau_{322}^{(0)} - \tau_{333}^{(0)};
\end{aligned}
\tag{23}
$$

which is solvable as a closed set, given initial values for the moments.

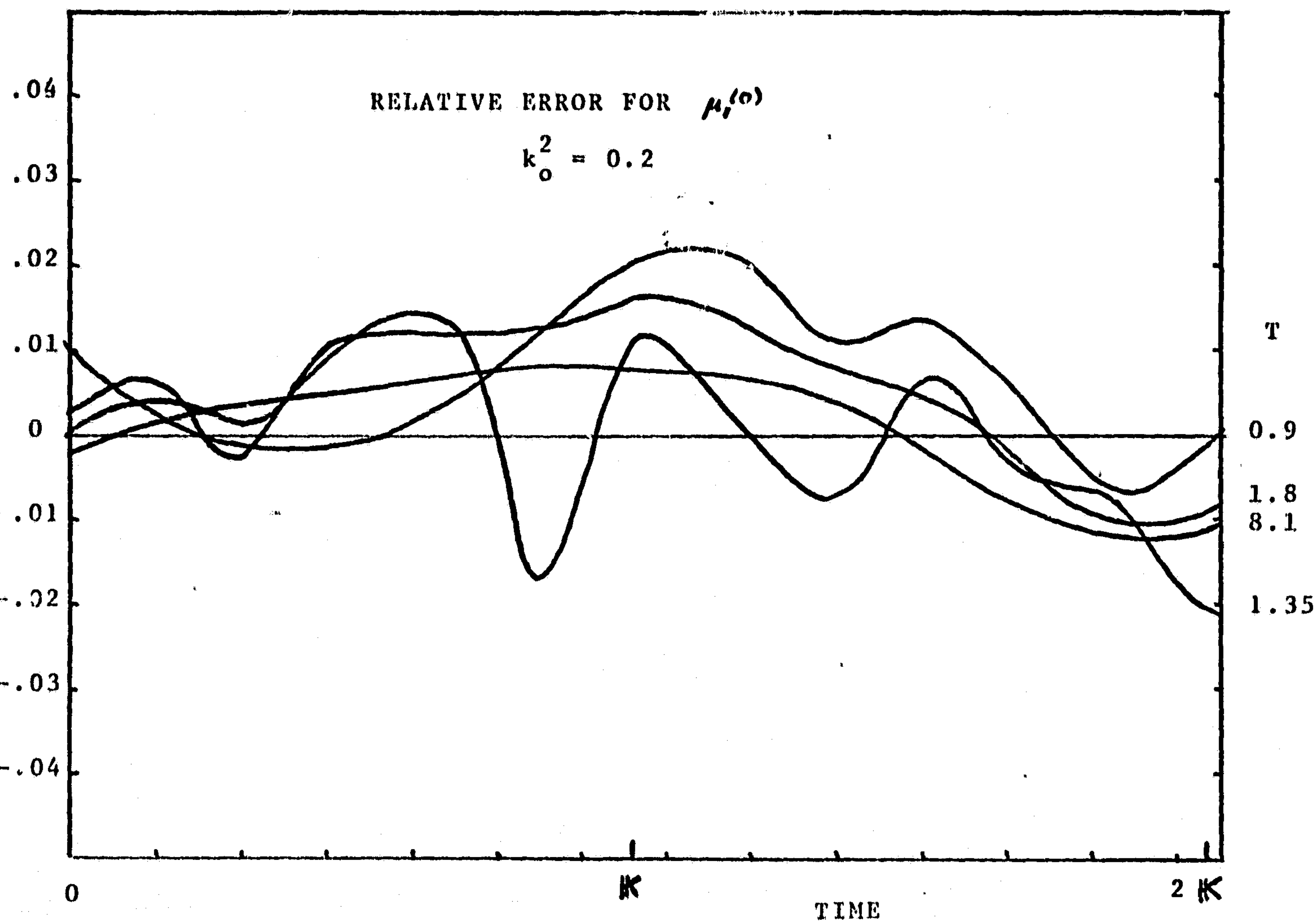
We note that to this order, we have no values, for example for the

covariances σ_{12}, σ_{13}, as well as for several of the higher order moments. To obtain estimates of the former, we need to go to the next higher order equations namely

$$
\begin{aligned}
\dot{\mu}_1^{(1)} &= -\sigma_{23}^{(0)} - \mu_2^{(0)}\mu_3^{(0)} \\
\dot{\sigma}_{12}^{(1)} &= -\mu_2^{(0)}\sigma_{23}^{(0)} + \mu_1^{(0)}\sigma_{13}^{(1)} - \mu_3^{(0)}\sigma_{22}^{(0)} - \tau_{223}^{(0)} \\
\dot{\sigma}_{13}^{(1)} &= -\mu_2^{(0)}\sigma_{33}^{(0)} - \mu_3^{(0)}\sigma_{23}^{(0)} - \mu_1^{(0)}\sigma_{12}^{(1)} - \tau_{233}^{(0)} \\
\dot{\mu}_2^{(1)} &= \mu_1^{(0)}\mu_3^{(1)} + \mu_3^{(0)}\mu_1^{(1)} + \sigma_{13}^{(1)} \\
\dot{\mu}_3^{(1)} &= -\mu_1^{(0)}\mu_2^{(1)} - \mu_2^{(0)}\mu_1^{(1)} - \sigma_{12}^{(1)} \\
\dot{\tau}_{123}^{(1)} &= F(\lambda_{2233}^{(0)}, \tau_{133}^{(1)}, \tau_{122}^{(1)}) \\
\dot{\tau}_{133}^{(1)} &= F(\lambda_{2333}^{(0)}, \tau_{123}^{(1)}) \\
\dot{\tau}_{122}^{(1)} &= F(\lambda_{2223}^{(0)}, \tau_{123}^{(1)}) \\
\dot{\sigma}_{22}^{(1)} &= F(\sigma_{23}^{(1)}, \tau_{123}^{(1)}) \\
\dot{\sigma}_{33}^{(1)} &= F(\sigma_{23}^{(1)}, \tau_{123}^{(1)}) \\
\dot{\sigma}_{23}^{(1)} &= F(\sigma_{33}^{(1)}, \sigma_{22}^{(1)}, \tau_{133}^{(1)}, \tau_{122}^{(1)})
\end{aligned}
\qquad (24)
$$

in which the F represent complicated functions of the indicated variables which we have no need to specify here. Equation (24) forms another closed set for a number of the higher order moments, as well as the two covariances σ_{12} and σ_{13} that were not obtained by solution of equation (23).

Earlier we presented some results on the error that results from the small k_o assumption (figure (1)) for the infinite time averaged case. Finite time averaging does not worsen the situation, as we illustrate in figure (2), which plots that variation in time (in units of the periodicity $\mathbb{K}$ of the elliptic function) of the error in $\mu_1^{(0)}$ for various values of the averaging time, T.



RELATIVE ERROR FOR $\mu_1^{(0)}$
$k_o^2 = 0.2$
.04
.03
.02
.01
0
-.01
-.02
-.03
-.04
T
0.9
1.8
8.1
1.35
0
K
2 K
TIME

FIGURE 2.

The fractional error rarely exceeds 2%.

The above illustrates the systematic procedure that can be used to develop solutions for the moment equations. Whether the expansion (22) is appropriate for equations more general than the minimum equations is not yet known, though its basis in the small k_o^2 assumption appears to have a logical physical foundation If indeed extendable, the result is of great importance, providing at first sight a means for dealing with the very formidable problem of closing the moment equation system that defines climate means and statistics.

Seasonal cycle simulation

The minimum equation system (1) is based on the use of the two-dimensional barotropic vorticity equations without forcing or dissipation:

$$\frac{D}{Dt}(\nabla \cdot \nabla \psi) = 0 \tag{25}$$

or

$$\frac{\partial}{\partial t}(\nabla^2\psi) - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = 0 \tag{26}$$

where ψ is the stream function. We introduce a forcing term

$$F(t) = F_o \sin(\Omega t) \tag{27}$$

where Ω corresponds to a seasonal cycle, on the right hand side of equation (25).

Fourier transformation of (25), when so modified, and retention of only three Fourier components, as with the development of the unforced minimum equations, yields the new set of equations:

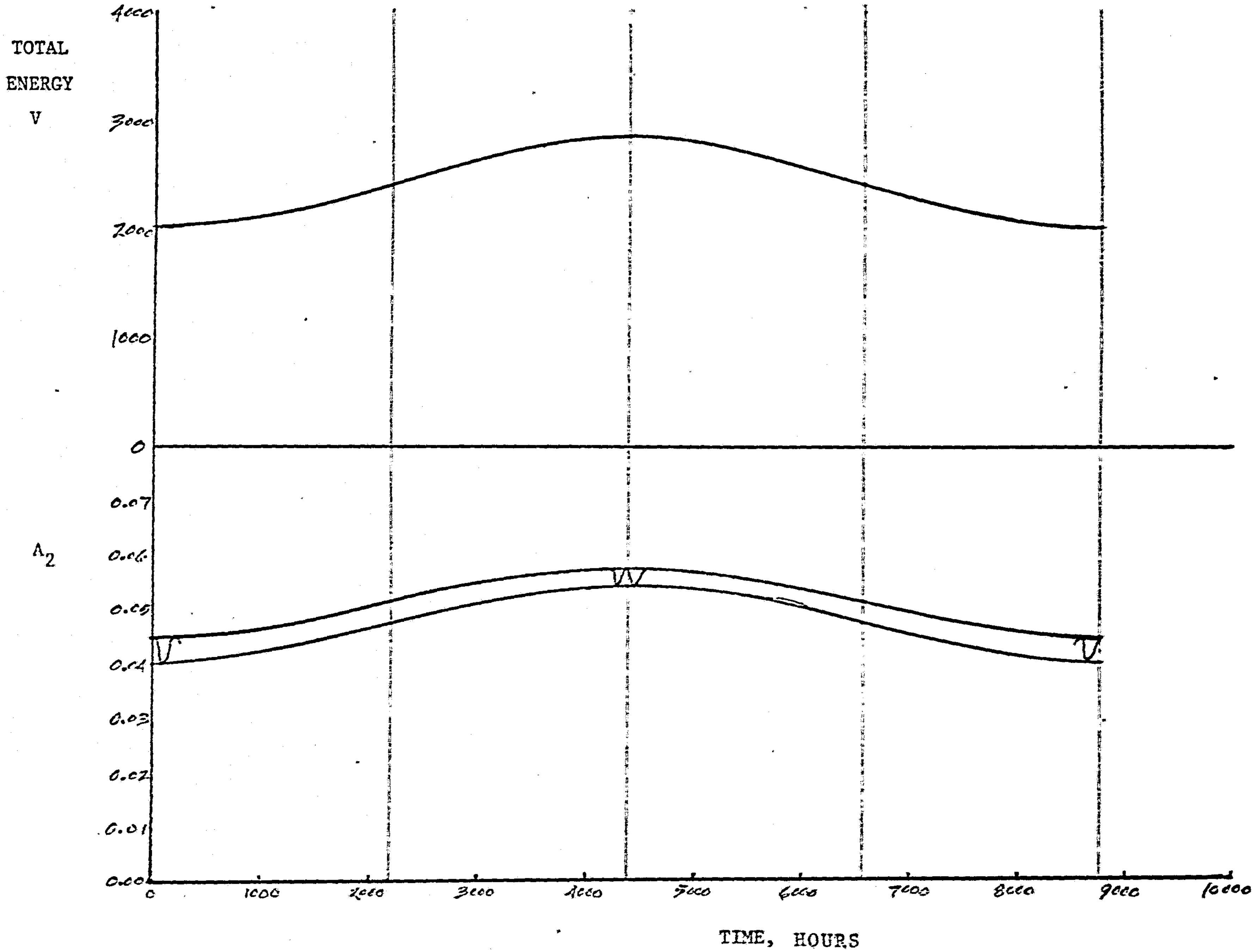
$$\frac{dA_1}{dt} = A_2 A_3 + F(t) \tag{28}$$

$$\frac{dA_2}{dt} = A_3 A_1 + F(t)$$

$$\frac{dA_3}{dt} = A_1 A_2 + \frac{1}{2} F(t)$$

for the Fourier components of the transformed vorticity, $\nabla^2\psi$. With $F(t) = 0$ the normal form of the minimum equations (1) is regained.

In contrast to the case $F(t) = 0$, analytic solution of (28) is not possible and numerical techniques have to be used. A numerical Taylor series expansion technique for solving (28) has been carried out for a range of values of the amplitude of forcing F_0 . Stability of the systems seems to be quite sensitive to the choice of F_o , but has not been investigated in detail.However, a choice of $F_o = 4.8\times10^{-6}$ sec^{-2} gives good stability as well as a seasonal amplitude variation that looks reasonable. Figure 4 is an example of some of the results. We show the variation during a year of the energy in the system and the behavior of the component A_2 (equation 28). The band of values for A_2 correspond to the limits of its oscillation amplitude at a high frequency time scale, corresponding to the unforced elliptic function behavior (equation 5). This frequency now varies during the season, in accordance with the formula (30) below; a total of approximately 150 cycles occur in one year. The frequency variation is in fact quite slow, according to the numerical results, as is the variation in amplitudes of the three components A_1 , A_2 and A_3 . To within the accuracy of the calculation, the system appears to return exactly to its initial state after a one year cycle. At least for the magnitude of forcing that we have used here, the effect of slow forcing appears to be rather simple and can be represented in quasi-stationary fashion by introducing appropriate values for the constants that appear in the unforced elliptic function solution (5) which we let to vary with the season.



TOTAL
ENERGY
V
4000
3000
2000
1000
0
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
A_2
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
TIME, HOURS

FIGURE 3.

From this we draw the important conclusion that imposition of forcing in our case does not lead to a major change in the spectral distribution of the solution, apart from the superposition of the slow amplitude variation for A_2. (There is no corresponding slow change in A_1 or A_3). In particular, there is no additional long period non-linear oscillation introduced that could interact with the forcing frequency.

An empirically determined form for the variation in amplitude and frequency of the elliptic function solutions induced by the forcing we find to be

$$\begin{aligned} A_1 &\simeq A_1^* \, sn\left[f(t)(ht-\kappa)\right] \\ A_2 &\simeq A_2^* \, f(t) \, dn\left[f(t)(ht-\kappa)\right] \\ A_3 &\simeq A_3^* \, cn\left[f(t)(ht-\kappa)\right] \end{aligned} \tag{29}$$

in which the modulus of the elliptic function k_o' is given by

$$k_o'^2 = k_o^2 / f^2(t) \tag{30}$$

where k_o' is the value with no forcing $(F(t) = 0)$, and

$$f(t) = 1 + \frac{1}{B^4} \int_0^t F(t')\,dt' \tag{31}$$

It seems that these results should be derivable analytically as approximations for small values of dF/dt. We have yet to show that this is the case.

Linearisation of the moment equations

Linearisation of averaged atmospheric state variables for the study of climate change should be a valid approximation for small perturbations of climate. Note that linearisation for a deterministic (non-averaged) computation of atmospheric dynamics is not valid for estimating long term (climatic) changes, because

of the large day-to-day variability of the state variables.

Our previous work on stochastic dynamic weather prediction[2] on a linearised approach to assessing the growth in error of weather forecasting models, suggested that an analagously useful result might hold in the consideration of small climatic changes. Thus we hoped that the statistical moments for small perturbations in climate would progressively decrease in magnitude as their order increased, thus leading to the possibility of a systematic closure procedure for the corresponding moment equations.

We have been carrying through a check on the validity of this idea using the minimum equations as a test case. We have developed numerical algorithms for truncation of linearised forms of the moment equations for the minimum equation set at the second and third orders. The accuracy of such approximations can be judged by comparison with exact solutions for the moments obtainable from the known analytic solutions of the minimum equations (equation 5).

Persistant programming errors have thus far prevented us for assessing whether this approach to closure of the moment equations is or is not possible.

<u>Future work</u>

Although some questions remain concerning the significance of the small k_o approximation discussed above, we believe we have now covered this particular property of the minimum equations adequately. We have yet to ascertain whether the important simplification in the stochastic formulation that it implies is extendible to other more complete and realistic climate simulation models.

A serious concern in planning our future studies relates to the relevance to the real world situation of the minimum equation behavior in their original or in modified form (modified by the introduction of a β effect, deterministic or stochastic forcing and dissipation). It is clear that the minimum equations

are vastly oversimplified in many respects, and their utility as partial simulators of real atmospheric effects depends a great deal of the mathematical richness of their non-linear properties. Let us briefly discuss two major apparent weaknesses. Firstly, we note that all three solutions (5) are phase coherent; but introduction of the Coriolis gradient effect, as demonstrated by Paegle and Robl[3], eliminates this oversimple property without destroying analytic simplicity. Secondly, the solutions of the minimum equations are strictly periodic, as therefore are also their climatic means. This is quite different from more complicated simulation models, such as that of Lorenz[4], as well as the real atmosphere. Non-periodicity is a fundamental characteristic for long term behavior of non-linear systems that it seems important for us to duplicate in any simplified models.

We had, until recently, considered the latter deficiency of the minimum equations to be serious enough to cause us to drop their use as simulators of even the most simple of climatic phenomena. However, a new paper by Dutton[5] has demonstrated the radical effect on behavior of the minimum equations when stochastic forcing is added. In particular, he shows that the revised system has bifurcation properties and multiple solutions, with a degree of complexity quite surprising in view of the simplicity of the behavior of the original, unforced system. In view of this result we have revised our opinion, and we now propose to continue study of the minimum equations, introducing stochastic forcing as we had originally planned, and now also studied by Dutton. In contrast to his work, however, which was a deterministic analysis of the time evolution of a single trajectory in phase space, we shall look at the stochastic properties of the forced system. This work has yet to be started, but we hope to do so in the coming year.

With the completion of the studies on seasonal forcing discussed above, and the proposed introduction of stochastic forcing, there only remains the introduction

of the β effect to round out a complete picture of what the three-component minimum equations system can yield. It is our present intent to try to carry through such a program to fruition.

We should also mention here yet another property of the minimum equations that has in the past led to criticism of their use as a real world simulator, but which recent developments in theory of large scale atmospheric simulation down-plays. The minimum equations describe an inherently two dimensional barotropic model for long term atmospheric behavior, thus omitting all the baroclinic effects that are essential in an understanding of the overall atmospheric circulation pattern and its energetics. We now see appearing, however, a growing body of theoretical and numerical work[6] [7] [8] suggesting that large scale atmospheric structures, not only on earth but even on Jupiter and Venus[9], can very well be simulated in barotropic models. Forcing of such models must, of course, arise from baroclinic effects, as indeed would be represented in our proposed further work with the minimum equations, by stochastic forcing terms.

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